

# VECTOR REARRANGEMENT INVARIANT BANACH SPACES OF RANDOM VARIABLES WITH EXPONENTIAL DECREASING TAILS OF DISTRIBUTIONS

OSTROVSKY E., SIROTA L.

Department of Mathematics and Statistics, Bar-Ilan University,  
59200, Ramat Gan, Israel.  
e-mail: eugostrovsky@list.ru

Department of Mathematics and Statistics, Bar-Ilan University,  
59200, Ramat Gan, Israel.  
e-mail: sirota3@bezeqint.net

ABSTRACT.

We present in this paper the theory of multivariate Banach spaces of random variables with exponential decreasing tails of distributions.

*Key words and phrases:* Random variable and random vector (r.v.), centered (mean zero) r.v., moment generating function, rearrangement invariant Banach space on vector random variables (vectors), ordinary and exponential moments, absolutely and relatively monotonic functions, multivariate Bernstein's theorem, Grand Lebesgue Space (GLS) norm, octant, tail of distribution, Young-Orlicz function, norm, Chernov's estimate, theorem of Fenchel-Morau, generating function, upper and lower estimates, non-asymptotical exponential estimations, Kramer's condition.

*Mathematics Subject Classification (2000):* primary 60G17; secondary 60E07; 60G70.

## 1 Introduction. Previous results.

We present here for beginning some known facts from the theory of one-dimensional random variables with exponential decreasing tails of distributions, see [14], [10], [13], chapters 1,2.

Especially we mention the authors preprint [15]; we offer in comparison with existing there results a more fine approach.

Let  $(\Omega, F, \mathbf{P})$  be a probability space,  $\Omega = \{\omega\}$ .

Let also  $\phi = \phi(\lambda), \lambda \in (-\lambda_0, \lambda_0), \lambda_0 = \text{const} \in (0, \infty]$  be certain even strong convex which takes positive values for positive arguments twice continuous differen-

tiabile function, such that

$$\phi(0) = \phi'(0) = 0, \quad \phi''(0) > 0, \quad \lim_{\lambda \rightarrow \lambda_0} \phi(\lambda)/\lambda = \infty. \quad (1.1)$$

We denote the set of all these (Young-Orlicz) function as  $\Phi$ ;  $\Phi = \{\phi(\cdot)\}$ .

We say by definition that the *centered* random variable (r.v)  $\xi = \xi(\omega)$  belongs to the space  $B(\phi)$ , if there exists some non-negative constant  $\tau \geq 0$  such that

$$\forall \lambda \in (-\lambda_0, \lambda_0) \Rightarrow \max_{\pm} \mathbf{E} \exp(\pm \lambda \xi) \leq \exp[\phi(\lambda \tau)]. \quad (1.2)$$

The minimal value  $\tau$  satisfying (1.2) for all the values  $\lambda \in (-\lambda_0, \lambda_0)$ , is named a  $B(\phi)$  norm of the variable  $\xi$ , write

$$\|\xi\|_{B(\phi)} \stackrel{def}{=}$$

$$\inf\{\tau, \tau > 0 : \forall \lambda : |\lambda| < \lambda_0 \Rightarrow \max_{\pm} \mathbf{E} \exp(\pm \lambda \xi) \leq \exp(\phi(\lambda \tau))\}. \quad (1.3)$$

These spaces are very convenient for the investigation of the r.v. having an exponential decreasing tail of distribution, for instance, for investigation of the limit theorem, the exponential bounds of distribution for sums of random variables, non-asymptotical properties, problem of continuous and weak compactness of random fields, study of Central Limit Theorem in the Banach space etc.

The space  $B(\phi)$  with respect to the norm  $\|\cdot\|_{B(\phi)}$  and ordinary algebraic operations is a Banach space which is isomorphic to the subspace consisted on all the centered variables of Orlicz's space  $(\Omega, F, \mathbf{P}), N(\cdot)$  with  $N$  – function

$$N(u) = \exp(\phi^*(u)) - 1, \quad \phi^*(u) = \sup_{\lambda} (\lambda u - \phi(\lambda)).$$

The transform  $\phi \rightarrow \phi^*$  is called Young-Fenchel transform. The proof of considered assertion used the properties of saddle-point method and theorem of Fenchel-Moraux:

$$\phi^{**} = \phi.$$

The next facts about the  $B(\phi)$  spaces are proved in [10], [13], p. 19-40:

$$1. \xi \in B(\phi) \Leftrightarrow \mathbf{E}\xi = 0, \text{ and } \exists C = \text{const} > 0,$$

$$U(\xi, x) \leq \exp(-\phi^*(Cx)), x \geq 0, \quad (1.4)$$

where  $U(\xi, x)$  denotes in this section the *one-dimensional tail* of distribution of the r.v.  $\xi$ :

$$U(\xi, x) = \max(\mathbf{P}(\xi > x), \mathbf{P}(\xi < -x)), \quad x \geq 0,$$

and this estimation is in general case asymptotically as  $x \rightarrow \infty$  exact.

Here and further  $C, C_j, C(i)$  will denote the non-essentially positive finite "constructive" constants.

More exactly, if  $\lambda_0 = \infty$ , then the following implication holds:

$$\lim_{\lambda \rightarrow \infty} \phi^{-1}(\log \mathbf{E} \exp(\lambda \xi)) / \lambda = K \in (0, \infty)$$

if and only if

$$\lim_{x \rightarrow \infty} (\phi^*)^{-1}(|\log U(\xi, x)|) / x = 1/K.$$

Hereafter  $f^{-1}(\cdot)$  denotes the inverse function to the function  $f$  on the left-side half-line  $(C, \infty)$ .

Let  $F = \{\xi(t)\}$ ,  $t \in T$ ,  $T$  is an arbitrary set, be the family of somehow dependent mean zero random variables. The function  $\phi(\cdot)$  may be constructive introduced by the formula

$$\phi(\lambda) = \phi_F(\lambda) \stackrel{def}{=} \max_{\pm} \log \sup_{t \in T} \mathbf{E} \exp(\pm \lambda \xi(t)), \quad (1.5)$$

if obviously the family  $F$  of the centered r.v.  $\{\xi(t), t \in T\}$  satisfies the so-called *uniform* Kramers condition:

$$\exists \mu \in (0, \infty), \sup_{t \in T} U(\xi(t), x) \leq \exp(-\mu x), x \geq 0.$$

In this case, i.e. in the case the choice the function  $\phi(\cdot)$  by the formula (1.5), we will call the function  $\phi(\lambda) = \phi_0(\lambda)$  a *natural* function.

**2.** We define  $\psi(p) = p/\phi^{-1}(p)$ ,  $p \geq 2$ .

Let us introduce a new norm, the so-called moment norm, or equally Grand Lebesgue Space (GLS) norm, on the set of r.v. defined in our probability space by the following way: the space  $G(\psi)$  consist, by definition, on all the centered (mean zero) r.v. with finite norm

$$\|\xi\| G(\psi) \stackrel{def}{=} \sup_{p \geq 1} [\|\xi\|_p / \psi(p)], \quad (1.6)$$

here and in what follows as ordinary

$$|\xi|_p := \mathbf{E}^{1/p} |\xi|^p = \left[ \int_{\Omega} |\xi(\omega)|^p \mathbf{P}(d\omega) \right]^{1/p}.$$

It is proved that the spaces  $B(\phi)$  and  $G(\psi)$  coincides:  $B(\phi) = G(\psi)$  (set equality) and both the norms  $\|\cdot\|_{B(\phi)}$  and  $\|\cdot\|$  are linear equivalent:  $\exists C_1 = C_1(\phi), C_2 = C_2(\phi) = \text{const} \in (0, \infty), \forall \xi \in B(\phi) \Rightarrow$

$$\|\xi\| G(\psi) \leq C_1 \|\xi\|_{B(\phi)} \leq C_2 \|\xi\| G(\psi). \quad (1.6a)$$

**3.** The definition (1.6) is correct still for the non-centered random variables  $\xi$ . If for some non-zero r.v.  $\xi$  we have  $\|\xi\| G(\psi) < \infty$ , then for all positive values  $u$

$$\mathbf{P}(|\xi| > u) \leq 2 \exp(-u/(C_3 \|\xi\|G(\psi))). \quad (1.7)$$

and conversely if a r.v.  $\xi$  satisfies (1.7), then  $\|\xi\|G(\psi) < \infty$ .

**We intend to extend in this report the definition and properties of these spaces into a multidimensional case, i.e. when all the considered r.v. are vectors.**

We agree to omit the proofs for the multivariate case if they are likewise to ones for the ordinary numerical variables.

The paper is organized as follows. In the second section we give necessary definitions, conventions and restrictions. We prove in the section 3 the Banach structure of offered spaces. The fourth section is devoted to the description of the characterization features for multidimensional moment generating function.

In the fifth section we investigate the linear transforms for vectors in these spaces. In the next section we obtain some multidimensional bilateral tail inequalities for random vectors belonging to these spaces. The seventh section contains states the equivalence with moments (Grand Lebesgue Spaces) and introduced norms.

We consider in the 8<sup>th</sup> one of the important applications of introduced spaces: the exponential bounds for the sums of independent random vectors. We discover in the penultimate section the relation between our spaces with multivariate Orlicz spaces.

The last section contains some concluding remarks.

## 2 Notations. Definitions, conventions and restrictions.

We need to introduce some new notations.

Denote by  $\epsilon = \vec{\epsilon} = \{\epsilon(1), \epsilon(2), \dots, \epsilon(d)\}$  the non-random  $d$  – dimensional numerical vector,  $d = 2, 3, \dots$ , whose components take the values  $\pm 1$  only.

Set  $\vec{1} = (1, 1, \dots, 1) \in R_+^d$ .

Denote by  $\Theta = \Theta(d) = \{\vec{\epsilon}\}$  collection of all such a vectors. Note that  $\text{card } \Theta = 2^d$  and  $\vec{1} \in \Theta$ .

Another denotations. For  $\vec{\epsilon} \in \Theta(d)$  and vector  $\vec{x}$  we introduce the coordinatewise product as a  $d$  – dimensional vector of the form

$$\vec{\epsilon} \otimes \vec{x} \stackrel{\text{def}}{=} \{\epsilon(1) x(1), \epsilon(2) x(2), \dots, \epsilon(d) x(d)\}.$$

Let  $f = f(\lambda)$ ,  $\lambda \in R^d$  be sufficiently smooths function and  $\vec{k}$  be non-negative  $d$  – dimensional integer vectors  $\vec{k} = \{k(1), k(2), \dots, k(d)\}$ ;  $|k| := \sum_j k(j)$ . We denote as ordinary

$$\frac{\partial^{\vec{k}} f}{\partial \vec{\lambda}^{\vec{k}}} = \frac{\partial^{|\vec{k}|} f(\lambda)}{\partial \lambda(1)^{k(1)} \partial \lambda(2)^{k(2)} \dots \partial \lambda(d)^{k(d)}}. \quad (2.0)$$

**Definition 2.1.**

Let  $\xi = \vec{\xi} = (\xi(1), \xi(2), \dots, \xi(d))$  be a centered random vector such that each its component  $\xi(j)$  satisfies the Kramer's condition. The *natural function*  $\phi_\xi = \phi_\xi(\lambda)$ ,  $\lambda = \vec{\lambda} = (\lambda(1), \lambda(2), \dots, \lambda(d)) \in R^d$  for the random vector  $\xi$  is defined as follows:

$$\exp\{\phi_\xi(\lambda)\} \stackrel{def}{=} \max_{\vec{\epsilon}} \mathbf{E} \exp \left\{ \sum_{j=1}^d \epsilon(j) \lambda(j) \xi(j) \right\} = \max_{\vec{\epsilon} \in \Theta} \mathbf{E} \exp \{ \epsilon(1) \lambda(1) \xi(1) + \epsilon(2) \lambda(2) \xi(2) + \dots + \epsilon(d) \lambda(d) \xi(d) \}, \quad (2.1)$$

where "max" is calculated over all the combinations of signs  $\epsilon(j) = \pm 1$ .

**Definition 2.2.**

The *tail function* for the random vector  $\vec{\xi} \sim U(\vec{\xi}, \vec{x})$ ,  $\vec{x} = (x(1), x(2), \dots, x(d))$ , where all the coordinates  $x(j)$  of the deterministic vector  $\vec{x}$  are non-negative, is defined as follows.

$$U(\vec{\xi}, \vec{x}) \stackrel{def}{=} \max_{\vec{\epsilon}} \mathbf{P} \left( \bigcap_{j=1}^d \{ \epsilon(j) \xi(j) > x(j) \} \right) = \max_{\vec{\epsilon} \in \Theta} \mathbf{P}(\epsilon(1) \xi(1) > x(1), \epsilon(2) \xi(2) > x(2), \dots, \epsilon(d) \xi(d) > x(d)), \quad (2.2)$$

where as before "max" is calculated over all the combinations of signs  $\epsilon(j) = \pm 1$ .

We illustrate this notion in the case  $d = 2$ . Let  $\vec{\xi} = (\xi(1), \xi(2))$  be a two-dimensional random vector and let  $x, y$  be non-negative numbers. Then

$$U((\xi(1), \xi(2)), (x, y)) = \max[\mathbf{P}(\xi(1) > x, \xi(2) > y), \mathbf{P}(\xi(1) > x, \xi(2) < -y), \mathbf{P}(\xi(1) < -x, \xi(2) > y), \mathbf{P}(\xi(1) < -x, \xi(2) < -y)].$$

**Definition 2.3.**

Let  $h = h(x)$ ,  $x \in R^d$  be some non-negative real valued function, which is finite on some non-empty neighborhood of origin. We denote as ordinary

$$\text{supp } h = \{x, h(x) < \infty\}.$$

The Young-Fenchel, or Legendre transform  $h^*(y)$ ,  $y \in R^d$  is defined likewise the one-dimensional case

$$h^*(y) \stackrel{def}{=} \sup_{x \in \text{supp } h} ((x, y) - h(x)). \quad (2.3)$$

Herewith  $(x, y)$  denotes the scalar product of the vectors  $x, y$  :  $(x, y) = \sum_j x(j)y(j)$ ;  $|x| = \sqrt{(x, x)}$ .

Obviously, if the set  $\text{supp } h$  is central symmetric, then the function  $h^*(y)$  is even.

**Definition 2.4.**

Recall, see [17], [18] that the function  $x \rightarrow g(x)$ ,  $x \in R^d$ ,  $g(x) \in R_+^1$  is named multivariate Young, or Young-Orlicz function, if it is even,  $d$  - times continuous differentiable, convex, non-negative, finite on the whole space  $R^d$ , and such that

$$\begin{aligned} g(x) = 0 &\Leftrightarrow x = 0; \quad \frac{\partial g}{\partial x} / (\vec{x} = 0) = 0, \\ \det \frac{\partial^2 g}{\partial x^2} / (\vec{x} = 0) &> 0. \end{aligned} \quad (2.4)$$

We explain in detail:

$$\frac{\partial g}{\partial x} = \left\{ \frac{\partial g}{\partial x_j} \right\} = \text{grad } g, \quad \frac{\partial^2 g}{\partial x^2} = \left\{ \frac{\partial^2 g}{\partial x_k \partial x_l} \right\}. \quad (2.5)$$

We assume finally

$$\lim_{|x| \rightarrow \infty} \frac{\partial^d g}{\prod_{k=1}^d \partial x_k} = \infty. \quad (2.6)$$

We will denote the set of all such a functions by  $Y = Y(R^d)$  and denote also by  $D$  introduced before matrix

$$D = D_g := \frac{1}{2} \left\{ \frac{\partial^2 g(0)}{\partial x_k \partial x_l} \right\}.$$

Evidently, the matrix  $D = D_g$  is non-negative definite, write  $D = D_g \geq 0$ .

**Definition 2.5.**

Let  $V$ ,  $V \subset R^d$  be open convex central symmetric:  $\forall x \in V \Rightarrow -x \in V$  subset of whole space  $R^d$  containing some non-empty neighborhood of origin. We will denote the collection of all such a sets by  $S = S(R^d)$ .

We will denote by  $Y(V)$ ,  $V \in S(R^d)$  the set of all such a functions from the definition 2.4 which are defined only on the set  $V$ . For instance, they are even, twice continuous differentiable, convex, non-negative, is equal to zero only at the origin and  $\det D_g > 0$ ,

$$\lim_{x \rightarrow \partial V - 0} \frac{\partial^d g}{\prod_{k=1}^d \partial x_k} = \infty. \quad (2.7)$$

Notation:  $V = \text{supp } g$ .

### 3 Definition and Banach structure of offered spaces.

#### Definition 3.1.

Let the set  $V$  be from the set  $S(R^d) : V \in S(R^d)$  and let the Young function  $\phi(\cdot)$  be from the set  $Y(V) : \text{supp } \phi = V$ .

We will say by definition likewise the one-dimensional case that the *centered* random vector (r.v)  $\xi = \xi(\omega) = \vec{\xi} = (\xi(1), \xi(2), \dots, \xi(d))$  with values in the space  $R^d$  belongs to the space  $B_V(\phi)$ , write  $\vec{\xi} \in B_V(\phi)$ , if there exists certain non-negative constant  $\tau \geq 0$  such that

$$\forall \lambda \in V \Rightarrow \max_{\vec{\epsilon}} \mathbf{E} \exp \left( \sum_{j=1}^d \epsilon(j) \lambda(j) \xi(j) \right) \leq \exp[\phi(\lambda \cdot \tau)]. \quad (3.1)$$

The minimal value  $\tau$  satisfying (3.1) for all the values  $\lambda \in V$ , is named by definition as a  $B_V(\phi)$  norm of the vector  $\xi$ , write

$$\|\xi\|_{B_V(\phi)} \stackrel{def}{=}$$

$$\inf \left\{ \tau, \tau > 0 : \forall \lambda : \lambda \in V \Rightarrow \max_{\vec{\epsilon}} \mathbf{E} \exp \left( \sum_{j=1}^d \epsilon(j) \lambda(j) \xi(j) \right) \leq \exp(\phi(\lambda \cdot \tau)) \right\}. \quad (3.2)$$

For example, the *generating function*  $\phi_\xi(\lambda)$  for these spaces may be picked by the following natural way:

$$\exp[\phi_\xi(\lambda)] \stackrel{def}{=} \max_{\vec{\epsilon} \in \Theta} \mathbf{E} \exp \left( \sum_{j=1}^d \epsilon(j) \lambda(j) \xi(j) \right), \quad (3.2a)$$

if of course the random vector  $\xi$  is centered and has an exponential tail of distribution. This imply that the natural function  $\phi_\xi(\lambda)$  is finite on some non-trivial central symmetry neighborhood of origin, or equivalently the mean zero random vector  $\xi$  satisfies the multivariate Kramer's condition.

Obviously, for the natural function  $\phi_\xi(\lambda)$

$$\|\xi\|_{B(\phi_\xi)} = 1.$$

It is easily to see that this choice of the generating function  $\phi_\xi$  is optimal, but in the practical using often this function can not be calculated in explicit view, but there is a possibility to estimate its.

We agree to take in the case when  $V = R^d$   $B_{R^d}(\phi) := B(\phi)$ .

Note that the expression for the norm  $\|\xi\|_{B_V(\phi)}$  dependent aside from the function  $\phi$  and the set  $V$ , only on the distribution  $\text{Law}(\xi)$ . Thus, this norm and

correspondent space  $B(\phi)$  are rearrangement invariant (symmetrical) in the terminology of the classical book [1], see chapters 1,2.

**Theorem 3.1.** *The space  $B_V(\phi)$  with respect to the norm  $\|\cdot\|_{B_V(\phi)}$  and ordinary algebraic operations is a rearrangement invariant vector Banach space.*

**Proof.** Let us prove at first the *triangle inequality*. We will denote for simplicity  $\|\xi\| = \|\xi\|_{B_V(\phi)}$  and correspondingly  $\|\eta\| = \|\eta\|_{B_V(\phi)}$ . It is reasonable to suppose  $\lambda \geq 0$ ,  $0 < \|\xi\|$ ,  $\|\eta\| < \infty$ .

Assume for definiteness  $\vec{\epsilon} = \vec{1} = (1, 1, \dots, 1)$ . We conclude on the basis of the direct definition of the  $B_V(\phi)$  norm for all the values  $\lambda \in V$

$$\mathbf{E}e^{(\lambda, \xi)} \leq e^{\phi(\|\xi\| \cdot \lambda)}, \quad \mathbf{E}e^{(\lambda, \eta)} \leq e^{\phi(\|\eta\| \cdot \lambda)}.$$

We apply the Hölder's inequality

$$\mathbf{E}e^{(\lambda, \xi + \eta)} \leq \sqrt[p]{e^{\phi(p \|\xi\| \cdot \lambda)}} \cdot \sqrt[q]{e^{\phi(q \|\eta\| \cdot \lambda)}}, \quad (3.3)$$

where as usually  $p, q = \text{const} > 1$ ,  $1/p + 1/q = 1$ . We can choose in the estimate (3.3)

$$p = \frac{\|\xi\| + \|\eta\|}{\|\xi\|}, \quad q = \frac{\|\xi\| + \|\eta\|}{\|\eta\|},$$

and obtain after substituting into (3.3)

$$\mathbf{E}e^{(\lambda, \xi + \eta)} \leq e^{\phi(\lambda \cdot (\|\xi\| + \|\eta\|))}, \quad (3.4)$$

therefore  $\|\xi + \eta\| \leq \|\xi\| + \|\eta\|$ .

The equality  $\|-\xi\| = \|\xi\|$  follows from the properties of parity of the function  $\phi(\cdot)$ , the equality  $\|\alpha \xi\| = \alpha \|\xi\|$ ,  $\alpha = \text{const} > 0$  follows from the direct definition (3.2) of norm in these spaces after simple calculations.

Finally, the *completeness* of our  $B_V(\phi)$  spaces follows immediately from the one-dimensional case as long as take place here the coordinatewise completeness.

**Remark 3.1.** In the article of Buldygin V.V. and Kozachenko Yu. V. [2] was considered a particular case when  $V = R^d$  and  $\phi(\lambda) = \phi^{(B)}(\lambda) = 0.5(B\lambda, \lambda)$ , where  $B$  is non-degenerate positive definite symmetrical matrix, as a direct generalization of the one-dimensional one notion, belonging to J.P.Kahane [9].

The correspondent random vector  $\vec{\xi}$  was named in [2] as a subgaussian r.v. relative the matrix  $B$  :

$$\forall \lambda \in R^d \Rightarrow \mathbf{E}e^{(\lambda, \xi)} \leq e^{0.5(B\lambda, \lambda) \|\xi\|^2}.$$

We will write in this case  $\xi \in \text{Sub}(B)$  or more precisely Law  $\xi \in \text{Sub}(B)$ .

**Remark 3.2.** Let the function  $\phi(\cdot) \in Y(V)$  be a given. Define the following operation between two non - negative numbers  $a$  and  $b$  :



$$a \odot b = a \odot_{\phi} b \stackrel{def}{=} \inf\{c, c > 0, \forall \lambda \in V \Rightarrow \phi(c \cdot \lambda) \geq \phi(a \cdot \lambda) + \phi(b \cdot \lambda)\}.$$

It is clear that  $0 \odot a = a$ ,  $a > 0$ ;

$$a \odot b = b \odot a, (a \odot b) \odot c = a \odot (b \odot c);$$

$$(\beta \cdot a) \odot (\beta \cdot b) = \beta \cdot (a \odot b), \beta = \text{const} \geq 0.$$

*We propose:* if both the random vectors  $\xi, \eta$  are independent and belongs to the space  $B_V(\phi)$ , then

$$\|\xi + \eta\| \leq \|\xi\| \odot_{\phi} \|\eta\|.$$

Indeed, by virtue of independence

$$\mathbf{E}e^{(\lambda, (\xi+\eta))} = \mathbf{E}e^{(\lambda, \xi)} \times \mathbf{E}e^{(\lambda, \eta)} \leq$$

$$e^{\phi(\|\xi\| \cdot \lambda)} \cdot e^{\phi(\|\eta\| \cdot \lambda)} \leq e^{\phi((\|\xi\| \odot \|\eta\|) \cdot \lambda)}.$$

As a slight consequence: if  $\{\xi_i\}$ ,  $i = 1, 2, \dots, n$  be a sequence of independent (centered) multidimensional r.v. to at the same space  $B(\phi)$ . Denote

$$S(n) = n^{-1/2} \sum_{i=1}^n \xi_i, \quad (3.5)$$

then

$$\|S(n)\|B(\phi) \leq n^{-1/2} \{ \|\xi_1\|B(\phi) \odot_{\phi} \|\xi_2\|B(\phi) \odot_{\phi} \dots \odot_{\phi} \|\xi_n\|B(\phi) \}. \quad (3.6)$$

**Remark 3.3.** Suppose the r.v.  $\xi$  belongs to the space  $B(\phi)$ ; then evidently  $\mathbf{E}\vec{\xi} = 0$ .

Suppose in addition that it has there the unit norm, then

$$\text{Var}(\vec{\xi}) \leq D_{\phi}; \quad \Leftrightarrow D_{\phi} \geq \text{Var}(\vec{\xi}). \quad (3.7)$$

It is interest to note that there exist many random vectors  $\eta = \vec{\eta}$  for which

$$\mathbf{E}e^{(\lambda, \eta)} \leq e^{0.5 (\text{Var}(\eta)\lambda, \lambda)}, \quad \lambda \in R^d, \quad (3.8)$$

see e.g. [3], [4], chapters 1,2; [13], p.53. V.V.Buldygin and Yu.V.Kozatchenko in [3] named these vectors *strictly subgaussian*; notation  $\xi \in \text{SSub}$  or equally  $\text{Law}(\xi) \in \text{SSub}$ .

V.V.Buldygin and Yu.V.Kozatchenko found also some interest applications of these notions.

## 4 Characterization features.

Statement of problem: given a function  $\phi(\cdot)$  from the set  $Y = Y(R^d)$ . Question: under what additional conditions there exists a mean zero random vector  $\xi$  which may be defined on the appropriate probability space such that

$$\forall \lambda \in R^d \Rightarrow \mathbf{E}e^{(\lambda, \xi)} = e^{\phi(\lambda)}, \quad (4.1)$$

or equally  $\phi(\lambda) = \phi_\xi(\lambda)$  for some random vector  $\xi$ .

It is clear that we hinted at multivariate Bernstein's theorem, see [5], [8]. Evidently,  $\phi(0) = 0$ .

Denote by  $\mu_\xi(\cdot)$  the (Borelian) distribution of r.v.  $\xi$ :  $\mu_\xi(A) := \mathbf{P}(\xi \in A)$ ,  $A \subset R^d$ , so that

$$\mathbf{E}e^{(\lambda, \xi)} = \int_{R^d} e^{(\lambda, x)} \mu_\xi(dx). \quad (4.2)$$

It will be presumed that the measure  $\mu_\xi$  and some ones have exponential decreasing tails, so that the integral in (4.2) converges for all the values  $\lambda \in R^d$ .

Let us consider the following integral

$$I_+(\lambda) := \int_{R_+^d} e^{(\lambda, x)} \mu_\xi(dx), \quad (4.3)$$

here and in the sequel

$$R_+^d \stackrel{def}{=} \{\vec{x} : \forall j \Rightarrow x(j) \geq 0\}.$$

We deduce

$$\frac{\partial^k I_+(\lambda)}{\partial \lambda(1)^{k(1)} \partial \lambda(2)^{k(2)} \dots \partial \lambda(d)^{k(d)}} = \int_{R_+^d} e^{(\lambda, x)} \cdot \prod_{j=1}^d x(j)^{k(j)} \cdot \mu_\xi(dx),$$

therefore

$$\frac{\partial^{\vec{k}} I_+(\lambda)}{\partial \vec{\lambda}^{\vec{k}}} = \frac{\partial^{|\vec{k}|} I_+(\lambda)}{\partial \lambda(1)^{k(1)} \partial \lambda(2)^{k(2)} \dots \partial \lambda(d)^{k(d)}} \geq 0, \quad (4.4)$$

where of course  $k, k(j) = 0, 1, 2, \dots$  and  $\sum_j k(j) = k$ .

The function satisfying the inequalities (4.4) for all the integer non - negative vectors  $\vec{k}$  are named *absolutely monotonic*.

The multivariate version of the classical Bernstein's theorem, see for example [8], tell us that if the function, say  $I(\lambda)$  is infinitely many times differentiable and satisfies the inequality (4.4) for all the integer positive vectors  $\vec{k}$ , then there exists a Borelian *measure*  $\nu$  on the set  $R_+^d$  such that

$$I(\lambda) = \int_{R_+^d} e^{(\lambda, x)} \nu(dx). \quad (4.5)$$

Let  $\vec{\epsilon} = \epsilon = \{\epsilon(1), \epsilon(2), \dots, \epsilon(d)\}$  be non-random element of the set  $\Theta$ . Recall that  $\vec{\epsilon}$  is  $d$  – dimensional numerical vector,  $d = 2, 3, \dots$ , whose components take the values  $\pm 1$  only. We associate for all such a vector  $\vec{\epsilon}$  the following octant in the whole space  $R^d$

$$Z(\vec{\epsilon}) = Z(\epsilon) = \{\vec{x} : \forall j \Rightarrow \epsilon(j) x(j) \geq 0\} \quad (4.6)$$

and correspondent integral

$$I(Z(\epsilon), \lambda) = \int_{Z(\epsilon)} e^{(\lambda, x)} \nu(dx). \quad (4.6a)$$

For instance,

$$Z(\vec{1}) = R_+^d; \quad I(Z(\vec{1}), \lambda) = \int_{R_+^d} e^{(\lambda, x)} \nu(dx) = I(\lambda).$$

We get alike (4.4)

$$\begin{aligned} \prod_{j=1}^d \epsilon(j)^{k(j)} &= \text{sign} \frac{\partial^{\vec{k}} I(Z(\vec{\epsilon}), \lambda)}{\partial \vec{\lambda}^{\vec{k}}} = \\ \text{sign} \frac{\partial^{|k|} I(Z(\vec{\epsilon}), \lambda)}{\partial \lambda(1)^{k(1)} \partial \lambda(2)^{k(2)} \dots \partial \lambda(d)^{k(d)}} &= \vec{\epsilon}^{\vec{k}}. \end{aligned} \quad (4.7)$$

**Definition 4.1.** Let the vector  $\vec{\epsilon} \in \Theta$  and related octant  $Z(\vec{\epsilon})$  be a given. The numerical function  $F = F(\lambda)$ ,  $\lambda \in R^d$  belongs by definition to the class  $K(\vec{\epsilon}) = K(\epsilon)$ , or on the other words, is monotonic relative the octant  $Z(\vec{\epsilon})$ , iff it satisfied the restriction (4.7) for all the values  $\vec{k}$ :

$$\forall \lambda \in R^d \Rightarrow \text{sign} \frac{\partial^{\vec{k}} F(\lambda)}{\partial \vec{\lambda}^{\vec{k}}} = \vec{\epsilon}^{\vec{k}}. \quad (4.8)$$

It follows immediately from the multivariate Bernstein's theorem after replacing  $x \rightarrow \epsilon \otimes x$  the following fact.

**Lemma 4.1.** The function  $F = F(\lambda)$ ,  $\lambda \in R^d$  belongs to the class  $K(\vec{\epsilon}) = K(\epsilon)$  iff it has a representation

$$F(\lambda) = \int_{Z(\epsilon)} e^{(\lambda, x)} \zeta(dx) \quad (4.9)$$

for some finite Borelian measure  $\zeta(\cdot)$ .

**Theorem 4.1.** The infinite differentiable function  $\phi(\lambda)$  such that  $\phi(0) = 0$  may be represented on the form (4.1):  $\phi = \phi_\xi$  for some random vector  $\xi = \vec{\xi}$  defined on some sufficiently rich probability space, iff the function  $\exp \phi(\lambda)$  has a representation of the form

$$\exp \phi(\lambda) = \sum_{\vec{\epsilon} \in \Theta} F_{\vec{\epsilon}}(\lambda), \quad (4.10)$$

where each function  $F_{\vec{\epsilon}}(\lambda)$  is monotonic relative the octant  $Z(\vec{\epsilon}) : F_{\vec{\epsilon}}(\cdot) \in K(\vec{\epsilon})$  and

$$\sum_{\vec{\epsilon} \in \Theta} F_{\vec{\epsilon}}(0) = 1. \quad (4.10a)$$

**Proof.** Assume first of all that the expression (4.10) take place. Denote by  $\chi_A(x)$  the indicator function of the measurable set  $A$ . We deduce using the proposition of lemma 4.1:

$$\begin{aligned} \sum_{\vec{\epsilon} \in \Theta} F_{\vec{\epsilon}}(\lambda) &= \sum_{\vec{\epsilon} \in \Theta} \int_{Z(\epsilon)} \exp(\lambda, x) \nu_{\epsilon}(dx) = \sum_{\vec{\epsilon} \in \Theta} \int_{R^d} \chi_{Z(\epsilon)}(x) \exp(\lambda, x) \nu_{\epsilon}(dx) = \\ &= \int_{R^d} \exp(\lambda, x) \sum_{\vec{\epsilon} \in \Theta} \chi_{Z(\epsilon)}(x) \nu_{\epsilon}(dx) = \int_{R^d} \exp(\lambda, x) \nu(dx), \end{aligned} \quad (4.11)$$

where

$$\nu(A) = \int_A \sum_{\vec{\epsilon} \in \Theta} \chi_{Z(\epsilon)}(x) \nu_{\epsilon}(dx). \quad (4.11a)$$

Substituting  $\lambda = 0$  into (4.11), we conclude that  $\nu$  is actually a probabilistic measure and

$$\exp \phi(\lambda) = \int_{R^d} \exp(\lambda, x) \nu(dx) = \mathbf{E} \exp(\lambda, \gamma), \quad (4.12)$$

where the r.v.  $\gamma$  has the distribution  $\nu$ .

Conversely, let

$$e^{\phi(\lambda)} = \mathbf{E} e^{(\lambda, \xi)}$$

for some  $d$  – dimensional random vector  $\xi$ , then

$$\begin{aligned} e^{\phi(\lambda)} &= \int_{R^d} e^{(\lambda, \xi)} \mu_{\xi}(dx) = \\ &= \sum_{\vec{\epsilon} \in \Theta} \int_{Z(\epsilon)} \exp(\lambda, x) \mu_{\epsilon}(dx) = \sum_{\vec{\epsilon} \in \Theta} F_{\vec{\epsilon}}(\lambda), \end{aligned}$$

where the functions  $F_{\vec{\epsilon}}(\lambda) :$

$$F_{\vec{\epsilon}}(\lambda) = \int_{Z(\epsilon)} \exp(\lambda, x) \mu_{\epsilon}(dx)$$

satisfy the conditions of lemma 4.1.

## 5 Linear transforms.

Statement of problem: given a  $d$  - dimensional (centered) random vector  $\xi$  from the space  $B(\phi)$ , in particular

$$\mathbf{E}e^{(\lambda, \xi)} \leq e^{\phi(\|\xi\| \cdot \lambda)}, \quad \phi \in Y(R^d), \quad (5.1)$$

and let  $A : R^d \rightarrow R^d$  be linear operator acting from the space  $R^d$  into itself. It is required to estimate some  $B(\tilde{\phi})$  norm of the linear transformed random vector  $\eta = A \cdot \xi$ .

Evidently, the r.v.  $\eta$  belongs to at the same space  $B(\phi)$ , as long as it may be represented as linear combination of the elements of these spaces. We intend to obtain more exact estimate.

Define the following function from at the same space  $\Phi$  :

$$\phi_A(\vec{\lambda}) := \phi(A^* \vec{\lambda}). \quad (5.2)$$

**Theorem 5.1.**

$$\|A\xi\|B(\phi) = \|\xi\|B(\phi_A). \quad (5.3)$$

**Proof** is very simple:  $\mathbf{E} \exp(\lambda, A\xi) =$

$$\mathbf{E} \exp(A^* \lambda, \xi) \leq \exp(\phi(\|\xi\| \cdot A^* \lambda)) = \exp(\phi_A(\|\xi\|)), \quad (5.4)$$

therefore

$$\|A\xi\|B(\phi) \leq \|\xi\|B(\phi_A).$$

Inverse inequality may be justify analogously.

**Example 5.1.**

Let  $\xi$  be subgaussian random vector relative certain positive definite symmetrical matrix  $R$ . Then

$$\|A\xi\| \text{Sub}(R) = \|\xi\| \text{Sub}(A R A^*). \quad (5.5)$$

We introduce now some generalization of the well - known notion of a  $\Delta_2$  condition on the multidimensional case.

**Definition 5.1.** We will say that the function  $\phi = \phi(\lambda), \lambda \in R^d$  satisfies the multidimensional  $\vec{\Delta}_2$  condition, write  $\phi(\cdot) \in \vec{\Delta}_2$ , if for arbitrary matrix  $A : R^d \rightarrow R^d$  there exists a non - negative constant, which we will denote by  $\|A\|_\phi$ , such that

$$\forall \lambda \in R^d \Rightarrow \phi(A^* \lambda) \leq \phi(\|A\|_\phi^2 \cdot \lambda). \quad (5.6)$$

It is easily to verify that the functional  $A \rightarrow |||A|||_\phi$  is actually certain matrix norm.

It follows from proposition of the example (5.1) that the quadratic form  $\lambda \rightarrow (R\lambda, \lambda)$  satisfies the  $\vec{\Delta}_2$  condition.

**Proposition 5.1.** It follows immediately from the relation (5.4) that if the function  $\phi \in Y(R^d)$  belongs also the class  $\vec{\Delta}_2$ , then under at the same notations and restrictions

$$||A\xi||B(\phi) \leq |||A|||_\phi \cdot ||\xi||B(\phi). \quad (5.7)$$

## 6 Tails behavior.

### A. Upper estimate.

Let  $\phi = \phi(\lambda)$ ,  $\lambda \in V \subset R^d$  be arbitrary non-negative real valued function, as in the definition 2.3, which is finite on some-empty neighborhood of origin. Suppose for given centered  $d$  – dimensional random vector  $\xi = \vec{\xi}$

$$\mathbf{E}e^{(\lambda, \xi)} \leq e^{\phi(\lambda)}, \quad \lambda \in V. \quad (6.1)$$

On the other words,  $||\xi||B(\phi) \leq 1$ .

**Proposition 6.1.** For all non-negative vector  $x = \vec{x}$  there holds

$$U(\vec{\xi}, \vec{x}) \leq \exp(-\phi^*(\vec{x})) - \quad (6.2)$$

the multidimensional generalization of Chernov's inequality.

**Proof.** Let for definiteness  $x_j > 0$ ; the case when  $\exists k \Rightarrow x_k \leq 0$  may be considered analogously.

We use the ordinary Tchebychev's inequality

$$U(\vec{\xi}, \vec{x}) \leq \frac{e^{\phi(\lambda)}}{e^{(\lambda, x)}} = e^{-(\lambda, x) + \phi(\lambda)}, \quad \lambda > 0. \quad (6.3)$$

Since the last inequality (6.3) is true for arbitrary non - negative vector  $\lambda \in V$ ,

$$U(\vec{\xi}, \vec{x}) \leq \inf_{\lambda \in V} e^{(\lambda, x) - \phi(\lambda)} = \exp \left\{ - \sup_{\lambda \in V} [ (\lambda, x) - \phi(\lambda) ] \right\} = \exp(-\phi^*(\vec{x})). \quad (6.4)$$

### B. Lower estimate.

**Theorem 6.1.** Let the function  $\phi(\cdot)$  be from the set  $S(R^d)$ , (Definition 2.5.) Suppose the mean zero random vector  $\xi = \vec{\xi}$  satisfies the condition (6.2) for all non-negative deterministic vector  $\vec{x} : \forall j = 1, 2, \dots, d \ x_j > 0$

$$U(\vec{\xi}, \vec{x}) \leq \exp(-\phi^*(\vec{x})). \quad (6.5)$$

We propose that r.v.  $\vec{\xi}$  belongs to the space  $B(\phi) : \exists C = C(\phi) \in (0, \infty)$ ,

$$\mathbf{E}e^{(\lambda, \xi)} \leq e^{\phi(C \cdot \lambda)}, \quad \lambda \in R^d. \quad (6.6)$$

**Proof** is likewise to the one-dimensional case, see [10], [13], p. 19-40; see also [15].

Note first of all that the estimate (6.6) is obviously satisfied for the values  $\lambda = \vec{\lambda}$  from the Euclidean unit ball of the space  $R^d : |\lambda| \leq 1$ , since the r.v.  $\xi$  is centered and has a very light (exponential decreasing) tail of distribution. It remains to consider further only the case when  $|\lambda| \geq 1$ .

We have using integration by parts

$$\mathbf{E}e^{(\lambda, \xi)} \leq \prod_{k=1}^d |\lambda_k| \cdot \int_{R^d} e^{(\lambda, x) - \phi^*(x)} dx \stackrel{def}{=} \prod_{k=1}^d |\lambda_k| \cdot I_{R^d}(\lambda).$$

It is sufficient to investigate the main part of the last integral, indeed

$$I_{R_+^d}(\lambda) := \int_{R_+^d} e^{(\lambda, x) - \phi^*(x)} dx. \quad (6.7)$$

Let  $\gamma = \text{const} \in (0, 1)$ . We have using Young inequality

$$(\lambda, x) = (\gamma\lambda, x/\gamma) \leq \phi^*(\gamma x) + \phi^{**}(\lambda/\gamma),$$

and after substituting into (6.7)

$$I_{R_+^d}(\lambda) \leq e^{\phi^{**}(\lambda/\gamma)} \cdot \int_{R_+^d} e^{\phi^*(\gamma x) - \phi^*(x)} dx = C_2(\gamma, \phi) \cdot e^{\phi^{**}(\lambda/\gamma)}. \quad (6.8)$$

We conclude by virtue of theorem Fenchel -Moraux  $\phi^{**}(\lambda) = \phi(\lambda)$ , therefore

$$\begin{aligned} \mathbf{E}e^{(\lambda, \xi)} &\leq C_3(\gamma, \phi) \left[ C_4(\gamma, \phi) + C_5(\gamma, \phi) \prod_{k=1}^d |\lambda_k| \right] \cdot e^{\phi(\lambda/\gamma)} \leq \\ &e^{\phi(\lambda/\gamma_2)}, \quad |\lambda| \geq 1, \end{aligned} \quad (6.9)$$

where  $\gamma_2 = \gamma_2(\phi) = \text{const} \in (0, 1)$ .

Another details are simple and may be omitted.

As a slight consequence:

**Corollary 6.1.** Let as above the function  $\phi(\cdot)$  be from the set  $S(R^d)$ , see the definition 2.5. The centered non-zero random vector  $\xi$  belongs to the space  $B(\phi) :$

$$\exists C_1 \in (0, \infty), \forall \lambda \in R^d \Rightarrow \mathbf{E}e^{(\lambda, \xi)} \leq e^{\phi(C_1 \cdot \lambda)}, \lambda \in R^d$$

if and only if

$$\exists C_2 \in (0, \infty), \forall x \in R_+^d \Rightarrow U(\vec{\xi}, \vec{x}) \leq \exp(-\phi^*(\vec{x}/C_2)).$$

More precisely, the following implication holds: there is finite positive constant  $C_3 = C_3(\phi)$  such that for arbitrary non-zero centered r.v.  $\xi : \|\xi\| = \|\xi\|B(\phi) < \infty \Leftrightarrow$

$$\forall \lambda \in R^d \Rightarrow \mathbf{E}e^{(\lambda, \xi)} \leq e^{\phi(\|\xi\| \cdot \lambda)}$$

iff

$$\exists C_3(\phi) \in (0, \infty) \forall x \in R_+^d \Rightarrow U(\vec{\xi}, \vec{x}) \leq \exp(-\phi^*(\vec{x}/(C_3/\|\xi\|))). \quad (6.10)$$

**Corollary 6.2.** Assume the non-zero centered random vector  $\xi = (\xi(1), \xi(2), \dots, \xi(d))$  belongs to the space  $B(\phi)$  :

$$\mathbf{E}e^{(\lambda, \xi)} \leq e^{\phi(\|\xi\| \cdot \lambda)}, \phi \in Y(R^d), \quad (6.11)$$

and let  $y$  be arbitrary positive non-random number. Then  $\forall y > 0 \Rightarrow$

$$\mathbf{P}\left(\min_{j=1,2,\dots,n} |\xi(j)| > y\right) \leq 2^d \cdot \exp(-\phi^*(y/\|\xi\|, y/\|\xi\|, \dots, y/\|\xi\|)). \quad (6.12)$$

**Example 6.1.** Let as before  $V = R^d$  and  $\phi(\lambda) = \phi^{(B)}(\lambda) = 0.5(B\lambda, \lambda)$ , where  $B$  is non-degenerate positive definite symmetrical matrix, in particular  $\det B > 0$ . It follows from theorem 6.1 that the (centered) random vector  $\xi$  is subgaussian relative the matrix  $B$  :

$$\forall \lambda \in R^d \Rightarrow \mathbf{E}e^{(\lambda, \xi)} \leq e^{0.5(B\lambda, \lambda)\|\xi\|^2}.$$

iff for some finite positive constant  $K = K(B, d)$  and for any non-random positive vector  $x = \vec{x}$

$$U(\xi, x) \leq e^{-0.5 \cdot ((B^{-1}x, x)/(K\|\xi\|^2))}. \quad (6.13)$$

## 7 Relation with moments.

We intend in this section to simplify the known proof the moment estimates (1.6)-(1.6a) for the one-dimensional r.v. and extend obtained result on the multivariate case.



We will use the following elementary inequality

$$x^r \leq \left(\frac{r}{\lambda e}\right)^r \cdot e^{\lambda x}, \quad r, \lambda, x > 0, \quad (7.0)$$

and hence

$$|x|^r \leq \left(\frac{r}{\lambda e}\right)^r \cosh(\lambda x), \quad r, \lambda > 0, \quad x \in R. \quad (7.0a)$$

As a consequence: let  $\xi$  be non-zero one-dimensional mean zero random variable belonging to the space  $B(\phi)$ . Then

$$\mathbf{E}|\xi|^r \leq 2 \left(\frac{r}{\lambda e}\right)^r e^{\phi(\lambda||\xi||)}, \quad \lambda > 0. \quad (7.1)$$

Author of the works [10], [13], chapters 1,2 choose in the inequality (7.1) the value

$$\lambda = \lambda_0 := \phi^{-1}(r/||\xi||)$$

and obtained the relations (1.6), (1.6a). We intend here to choose the value  $\lambda$  for reasons of optimality.

We hope that this method is more simple and allows easy multivariate generalization.

In detail, introduce the function

$$\Phi(\mu) := \phi(e^\mu), \quad \mu \in R. \quad (7.2)$$

Let temporarily for simplicity in (7.1) be  $||\xi|| = ||\xi||B(\phi) = 1$ . One can rewrite (7.1) as follows.

$$\begin{aligned} \mathbf{E}|\xi|^r &\leq 2 r^r e^{-r} e^{-r \ln \lambda + \phi(\lambda)} = \\ &2 r^r e^{-r} e^{-r\mu + \phi(e^\mu)} = 2 r^r e^{-r} e^{-(r\mu - \Phi(\mu))}, \end{aligned} \quad (7.3)$$

and we deduce after minimization over  $\mu$  or  $(\lambda)$

$$\mathbf{E}|\xi|^r \leq 2 r^r e^{-r} e^{-\Phi^*(r)}.$$

So, we proved in fact the following statement.

**Proposition 7.1.** Let  $\phi(\cdot)$  be arbitrary non - negative continuous function and let the centered numerical r.v.  $\xi$  be such that  $\xi \in B(\phi)$  or equally

$$U(\xi, x) \leq \exp(-\phi^*(x)), \quad x \geq 0.$$

Then

$$|\xi|_r \leq 2^{1/r} r e^{-1} e^{-\Phi^*(r)/r} ||\xi||B(\phi), \quad r > 0. \quad (7.4)$$

We want investigate now the inverse conclusion. Namely, let the mean zero numerical r.v.  $\xi$  be such that

$$|\xi|_r \leq C_0 r e^{-\Phi^*(r)/r}. \quad (7.5)$$

Let the function  $\Phi(\cdot)$  be from the set  $S(R)$  and suppose the real valued centered r.v.  $\xi$  satisfies the estimate (7.5) at least for the integer even values  $r = 2m$ ,  $m = 1, 2, \dots$ :

$$|\xi|_{2m} \leq K \cdot (2m) \cdot e^{-\Phi^*(2m)/(2m)}, \quad K = \text{const} \in (0, \infty). \quad (7.6)$$

The last inequality may be rewritten on the language of the  $G\psi$  spaces as follows. Define the correspondent  $\psi$  - function

$$\psi(m) = \psi_\phi(m) = (2m) \cdot e^{-\Phi^*(2m)/(2m)}, \quad m = 1, 2, \dots;$$

then the constant  $K$  in (7.6) has a form

$$K = \|\xi\| G\psi_\phi = \sup_m \left[ \frac{|\xi|_{2m}}{\psi_\phi(m)} \right]. \quad (7.7)$$

**Theorem 7.1.** Suppose  $\phi \in S(R^1)$ . Let  $\mathbf{E}\xi = 0$  and  $K = \|\xi\| G\psi_\phi < \infty$ . Then  $\xi \in B(\phi)$  and of course

$$\|\xi\| B(\phi) \leq C_1 \|\xi\| G\psi_\phi \leq C_2 \|\xi\| B(\phi), \quad 0 < C_1 = \text{const} \leq C_2 < \infty. \quad (7.8)$$

**Proof.** It remains to prove only the left-hand side of the bilateral inequality (7.8). We can assume without loss of generality  $\mathbf{E}\xi = 0$  and

$$|\xi|_{2m} \leq (2m) \cdot e^{-\Phi^*(2m)/(2m)}. \quad (7.9)$$

Let us consider the so - called moment generating function  $g(\lambda) = g_\xi(\lambda)$  for the r.v.  $\xi$ :

$$g(\lambda) = g_\xi(\lambda) \stackrel{\text{def}}{=} \mathbf{E}e^{\lambda\xi}. \quad (7.10)$$

Our target here is equivalent to justify the key estimate:  $\exists = C(m) \in (0, \infty)$  such that

$$g_\xi(\lambda) = \mathbf{E}e^{\lambda\xi} \leq e^{\phi(C(m)\lambda)}, \quad \lambda \in R. \quad (7.11)$$

The last inequality (7.11) is evident for the "small" values  $\lambda$ , say for  $|\lambda| \leq 1$ .

Let now  $|\lambda| \geq 1$ , for definiteness let  $\lambda \geq 1$ . We deduce taking into account the centeredness of the r.v.  $\xi$ :

$$\mathbf{E} \cosh \lambda\xi = 1 + \sum_{m=1}^{\infty} \frac{\lambda^{2m}}{(2m)!} \cdot \mathbf{E}\xi^{2m} \leq$$

$$1 + \sum_{m=1}^{\infty} \frac{C^m \lambda^{2m}}{(2m)!} \cdot (2m)^{2m} \cdot e^{-\Phi^*(2m)} \leq 1 + \sum_{m=1}^{\infty} C_2^m \cdot \lambda^{2m} \cdot e^{-\Phi^*(2m)}.$$

We apply the so-called discrete analog of the saddle - point method, see [19], chapter 3, sections 3, 4:

$$\begin{aligned} \mathbf{E} \cosh \lambda \xi &\leq \exp \left( \sup_{z \geq 1} [z \cdot \ln(C_3 \lambda) - \Phi^*(z)] \right) = \\ \exp(\Phi^{**}(\ln C_4 \lambda)) &= \exp(\Phi(\ln C_4 \lambda)) = \exp \phi(C_4 \lambda) \end{aligned} \quad (7.12)$$

again by virtue of theorem Fenchel-Morau.

This completes the proof of theorem 7.1.

**Example 7.1.** Suppose that the function  $\phi(\cdot)$  is from the set  $S(R^1)$  be such that for some constant  $p > 1$

$$\phi(\lambda) = \phi_p(\lambda) \leq C_1 |\lambda|^p, \quad |\lambda| > 1.$$

Let also the centered non-zero random variable  $\xi$  belongs to the space  $G\psi_p$ . Then

$$|\xi|_r \leq C_2(p) r^{1/q} \|\xi\| B(\phi_r), \quad r \geq 1, \quad q = p/(p-1),$$

and the inverse conclusion is also true: if  $\mathbf{E}\xi = 0$  and if for some constant  $K$

$$\forall r \geq 1 \Rightarrow |\xi|_r \leq K r^{1/q}, \quad q = p/(p-1),$$

then  $\xi(\cdot) \in B(\phi_p)$  and wherein  $\|\xi\| B(\phi_p) \leq C_3 K$ .

**Example 7.2.** Suppose now that the function  $\phi(\cdot)$  is from the set  $S(-K, K)$ ,  $0 < K = \text{const} < \infty$  be such that

$$\phi(\lambda) = \phi^{(K)}(\lambda) \leq \frac{C_4}{K - |\lambda|}, \quad |\lambda| < K.$$

Let also the centered non - zero random variable  $\xi$  belongs to this space  $G\psi^{(K)}$ . Then

$$|\xi|_r \leq C_5(K) r \|\xi\| B(\phi^{(K)}),$$

and likewise the inverse conclusion is also true: if  $\mathbf{E}\xi = 0$  and if for some finite positive constant  $K$

$$\forall r \geq 1 \Rightarrow |\xi|_r \leq K r,$$

then  $\xi(\cdot) \in B(\phi^{(K)})$  and herewith  $\|\xi\| B(\phi^{(K)}) \leq C_6 K$ .

We need getting to the presentation of the multidimensional case to extend our notations and restrictions. In what follows in this section the variables  $\lambda, r, x, \xi$  are as before vectors from the space  $R^d$ ,  $d = 2, 3, \dots$ , and besides  $r = \vec{r} = \{r(1), r(2), \dots, r(d)\}$ ,  $r(j) \geq 1$ .

Vector notations:

$$|r| = |\vec{r}| = \sum_j r(j), \quad |\xi| = |\vec{\xi}| = \{|\xi(1)|, |\xi(2)|, \dots, |\xi(d)|\} \in R_+^d,$$

$$\vec{x} \geq 0 \Leftrightarrow \forall j \quad x(j) \geq 0;$$

$$x^r = \vec{x}^{\vec{r}} = \prod_{j=1}^d x(j)^{r(j)}, \quad \vec{x} \geq 0,$$

$$\ln \vec{\lambda} = \{\ln \lambda(1), \ln \lambda(2), \dots, \ln \lambda(d)\}, \quad \vec{\lambda} > 0,$$

$$e^{\vec{\mu}} = \{e^{\mu(1)}, e^{\mu(2)}, \dots, e^{\mu(d)}\},$$

$$\Phi(\mu) = \Phi(\vec{\mu}) = \phi(e^{\vec{\mu}}),$$

$$\frac{r}{\lambda \cdot e} = \frac{\vec{r}}{\vec{\lambda} \cdot e} = \prod_{j=1}^d \left( \frac{r(j)}{e \lambda(j)} \right) = e^{-|r|} \cdot \prod_{j=1}^d \left( \frac{r(j)}{\lambda(j)} \right),$$

$$|\xi|_r = |\vec{\xi}|_{\vec{r}} = \left( \mathbf{E} |\vec{\xi}|^{\vec{r}} \right)^{1/|r|}.$$

We will use now the following elementary inequality

$$x^r \leq \left( \frac{r}{\lambda \cdot e} \right)^r \cdot e^{(\lambda, x)}, \quad r, \lambda, x > 0. \quad (7.13)$$

As a consequence: let  $\xi$  be non - zero  $d$ - dimensional mean zero random vector belonging to the space  $B(\phi)$ . Then

$$\mathbf{E} |\xi|^r \leq 2^d \left( \frac{r}{\lambda \cdot e} \right)^r e^{\phi(\lambda \|\xi\|)} = 2^d e^{-|r|} r^r \lambda^{-r} e^{\phi(\lambda \|\xi\|)}, \quad \lambda > 0. \quad (7.14)$$

We find likewise the one-dimensional case:

**Proposition 7.2.** Let  $\phi(\cdot)$  be arbitrary non - negative continuous function and let the centered numerical r.v.  $\xi$  be such that  $\xi \in B(\phi) : 0 < \|\xi\| = \|\xi\| B(\phi) < \infty$ . Then

$$|\vec{\xi}|_{\vec{r}} \leq e^{-1} \cdot 2^{d/|r|} \cdot \prod_j r(j)^{r(j)/|r|} \cdot e^{-\Phi^*(r)/|r|} \cdot \|\xi\| B(\phi), \quad r = \vec{r} > 0. \quad (7.15)$$

Note that in general case the expression  $|\xi|_r$  does not represent the norm relative the random vector  $\vec{\xi}$ .

But if we denote

$$\psi_\Phi(\vec{r}) := e^{-1} \cdot 2^{d/|r|} \cdot \prod_j r(j)^{r(j)/|r|} \cdot e^{-\Phi^*(r)/|r|}$$

and define

$$\|\xi\|G\psi_\Phi \stackrel{def}{=} \sup_{\vec{r} \geq 1} \left[ \frac{|\vec{\xi}|_{\vec{r}}}{\psi_\Phi(\vec{r})} \right],$$

we obtain some modification of the one-dimensional Grand Lebesgue Space (GLS) norm, see (1.6).

The statement of proposition 7.2 may be rewritten as follows.

$$\|\xi\|G\psi_\Phi \leq \|\xi\|B(\phi). \quad (7.15a)$$

Let us state the inverse up to multiplicative constant inequality.

**Theorem 7.2.** Suppose  $\phi \in S(R^d)$ . Let  $\mathbf{E}\xi = 0$  and  $K = \|\xi\|G\psi_\Phi < \infty$ . Then  $\xi \in B(\phi)$  and moreover both the norms  $\|\xi\|B(\phi)$  and  $\|\xi\|G\psi_\Phi$  are equivalent:

$$\|\xi\|B(\phi) \leq C_3(\phi) \|\xi\|G\psi_\Phi, \quad C_3(\phi) \in (0, \infty). \quad (7.16)$$

The proof of this theorem is at the same as one in the theorem 7.1 and may be omitted.

## 8 Exponential bounds for the sums of random vectors.

Statement of problem: given independent centered  $d$  – dimensional random vectors  $\xi_i$ ,  $i = 1, 2, \dots$ ; put as above

$$S(n) = n^{-1/2} \sum_{i=1}^n \xi_i,$$

the classical norming for the sum of independent mean zero i.d. random vectors.

It is required to deduce the exponential estimate for tails  $U(S(n), \vec{x})$ , (non-uniform estimates), as well as for uniform tails  $\sup_n U(S(n), \vec{x})$ , for sufficiently greatest values  $\vec{x}$ .

We claim to obtain an upper as well as lower exponential decreasing bounds for mentioned probabilities.

Of course, these estimates may be used in statistics and in the Monte - Carlo method, see [6], [7], namely, in the so-called "method of depending trials".

Related previous works: [16], [12], [13], chapters 2,3; [15].

### A. Upper estimates.

**Definition 8.1.** The function  $\phi : R^d \rightarrow R$  from the set  $Y = Y(R^d)$  belongs by definition to the class  $\Lambda_2$ ,  $\phi(\cdot) \in \Lambda_2$ , iff for all positive numbers  $a, b > 0$  and for all the vectors  $\lambda \in R^d$

$$\phi(a \cdot \lambda) + \phi(b \cdot \lambda) \leq \phi(\sqrt{a^2 + b^2} \cdot \lambda). \quad (8.1)$$

This condition can be rewritten as follows:  $\forall c = \text{const} > 0$  and for all the vectors  $\lambda \in R^d \Rightarrow$

$$\phi(\lambda) + \phi(c \cdot \lambda) \leq \phi(\sqrt{c^2 + 1} \cdot \lambda) \quad (8.1a)$$

or also as follows:  $\forall \theta \in [0, \pi/2]$  and for all the vectors  $\lambda \in R^d \Rightarrow$

$$\phi(\lambda \cos(\theta)) + \phi(\lambda \sin(\theta)) \leq \phi(\lambda). \quad (8.1b)$$

The condition (8.1) is trivially satisfied if for example

$$\phi(\lambda) = (Q\lambda, \lambda),$$

where  $Q$  is symmetrical positive definite matrix of a size  $d \times d$ .

**Lemma 8.1.** Assume that the function  $\phi(\cdot)$  has a form

$$\phi(\lambda) = \nu((Q\lambda, \lambda)), \quad (8.2)$$

where  $Q$  is symmetrical positive definite matrix of a size  $d \times d$ ,  $\nu : R^1 \rightarrow R^1$  is continuous numerical non - negative convex function such that  $\forall z \geq 0 \nu(z) = 0 \Leftrightarrow z = 0$ .

We propose that  $\phi(\cdot) \in \Lambda_2$ .

**An example:**

$$\phi(\lambda) = \nu(|\lambda|^2), \quad (8.3)$$

where the properties of the function  $\nu(\cdot)$  was described before.

The functions of the form (8.3) are named spherical, or radial.

**Proof.** We will use the following elementary for such a convex functions  $\{\nu(\cdot)\}$

$$\nu(x) + \nu(y) \leq \nu(x + y), \quad x, y \geq 0,$$

see [11], chapter 1, sections 1,2. Therefore

$$\nu(a^2(Q\lambda, \lambda)) + \nu(b^2(Q\lambda, \lambda)) \leq \nu((a^2 + b^2) \cdot (Q\lambda, \lambda)),$$

which is equivalent to the inequality (8.1).

**Theorem 8.1.** *Suppose the function  $\phi = \phi(\lambda)$  belongs to the set  $\Lambda_2$ . If  $\{\xi_i\}$ ,  $i = 1, 2, \dots, n$  are independent random vectors belonging to the space  $B(\phi)$  with finite norms in this space  $\|\xi_i\|_{B(\phi)} = \|\xi_i\|$ . We propose*

$$\|S(n)\|_{B(\phi)} \leq n^{-1/2} \left[ \sum_{i=1}^n \|\xi_i\|^2 \right]^{1/2} \stackrel{\text{def}}{=} \sigma(n). \quad (8.4)$$

**Proof** follows immediately from the estimate

$$\|\xi + \eta\| \leq \|\xi\| \odot_\phi \|\eta\| \leq \sqrt{\|\xi\|^2 + \|\eta\|^2}, \quad (8.5)$$

which take place for independent random vectors  $\xi, \eta$  from the space  $B(\phi)$ , (Pythagoras inequality), and from the homogeneity of the norm  $\|\xi\|_{B(\phi)}$ .

It follows immediately from the last inequality (6.5) and theorem 6.1

**Corollary 8.1.** *We affirm under conditions of theorem 8.1*

$$U(S(n), \vec{x}) \leq \exp(-\phi^*(\vec{x}/\sigma(n))), \quad \vec{x} \geq 0. \quad (8.6)$$

*If in addition  $\sigma := \sup_n \sigma(n) < \infty$ , then of course*

$$\sup_n U(S(n), \vec{x}) \leq \exp(-\phi^*(\vec{x}/\sigma)). \quad (8.6a)$$

Another approach. Let the r.v.  $\vec{\xi} = \xi$  be from some space  $B(\phi)$ ,  $\phi \in Y(V)$ . For instance, the function  $\phi(\cdot)$  may be the natural function for the r.v.  $\xi$ :  $\phi(\lambda) = \phi_\xi(\lambda)$ . Introduce a new functions

$$\phi_n(\lambda) \stackrel{\text{def}}{=} n \phi(\lambda/\sqrt{n}), \quad \bar{\phi}(\lambda) \stackrel{\text{def}}{=} \sup_n [n \phi(\lambda/\sqrt{n})]. \quad (8.7)$$

The last function  $\bar{\phi}(\lambda)$ ,  $\lambda \in \Lambda$  is finite as long as there exists a limit

$$\lim_{n \rightarrow \infty} n \phi(\lambda/\sqrt{n}) = 0.5 (\phi''(0)\lambda, \lambda) < \infty.$$

Let us explain the sense of this function. Suppose here the random vectors  $\{\xi_i\}$ ,  $i = 1, 2, \dots, n$  are independent identically distributed copies of  $\xi$ , then

$$\mathbf{E} e^{\lambda S(n)} \leq e^{\phi_n(\lambda)}, \quad (8.8)$$

and correspondingly

$$\sup_n \mathbf{E} e^{\lambda S(n)} \leq e^{\bar{\phi}(\lambda)}. \quad (8.8a)$$

**Proposition 8.1.** We conclude alike the theorem 8.1 applying theorem 6.1 under formulated in this pilcrow conditions

$$U(S(n), \vec{x}) \leq \exp(-\phi_n^*(\vec{x})), \vec{x} \geq 0. \quad (8.9)$$

$$\sup_n U(S(n), \vec{x}) \leq \exp(-\overline{\phi}^*(\vec{x})), \vec{x} \geq 0. \quad (8.9a)$$

## B. Lower bounds.

Lower estimates for the considered here probabilities are simple; we follow [13], p. 50 - 53, where it was considered the one-dimensional case  $d = 1$ .

Namely, let here  $\xi, \{\xi_i\}, i = 1, 2, \dots$  be a sequence of centered identical distributed independent random vectors having exponential decreasing tails, i.e. belonging to some space  $B(\phi)$ , where  $\phi(\cdot) \in S(R^d)$ . Then we have on the one hand

$$\sup_n U(S(n), \vec{x}) \geq U(\vec{\xi}, \vec{x}). \quad (8.10)$$

On the one hand,

$$\sup_n U(S(n), \vec{x}) \geq \lim_{n \rightarrow \infty} U(S(n), \vec{x}). \quad (8.11)$$

Let us denote  $Q = \text{Var}(\xi)$  and by  $\gamma_Q(\cdot)$  the Gaussian distribution (measure) on the space  $R^d$  with mean zero and variance  $Q$ . Then by virtue of multivariate CLT

$$\lim_{n \rightarrow \infty} U(S(n), \vec{x}) = \int_{\vec{y} \geq \vec{x}} \gamma_Q(dy). \quad (8.12)$$

The right - hand side of the relation (8.12) may be estimated as follows:

$$\int_{\vec{y} \geq \vec{x}} \gamma_Q(dy) \geq \exp(-C(Q) |x|^2), |x|^2 = (x, x) \geq 1.$$

We established in fact the following lower estimate.

### Proposition 8.2.

$$\sup_n U(S(n), \vec{x}) \geq \max\left(U(\vec{\xi}, \vec{x}), \exp(-C(Q) |x|^2)\right), |x| \geq 1. \quad (8.13)$$

**Example 8.1.** Let as before  $\xi, \{\xi_i\}, i = 1, 2, \dots$  be a sequence of centered identical distributed independent random vectors such that

$$U(\vec{\xi}, \vec{x}) \leq e^{-|x|^p}, |x| \geq 1,$$

where  $p = \text{const} \geq 1$ . Then

$$\sup_n U(S(n), \vec{x}) \leq \exp(-C(d, p) |\vec{x}|^{\min(p, 2)}), |x| \geq 1.$$

**Remark 8.1.** Note that the last estimate can not be improvable still in the one-dimensional case, see [10], [13], p. 50-53.



**Remark 8.2.** Note finally that the last estimate can not be obtained by means of Rosenthal's moment inequality, although the  $B(\phi)$  and moment (GLS) norm are (linear) equivalent.

This is due to the fact that the famous Rosenthal's "constants"  $R(p)$  tending to infinity as  $p \rightarrow \infty$  :

$$R(p) \asymp \frac{p}{\log p}, \quad p \in [2, \infty).$$

The detail explanation of this phenomenon may be found in [13], chapter 2, sections 2.1 and 2.3.

## 9 Relation with multivariate Orlicz spaces.

Let now the function  $\phi = \phi(\lambda)$  be from the set  $Y(V)$ , see definition 2.5. Introduce the following Young-Orlicz function

$$N_\phi(u) = N_\phi(\vec{u}) = \exp(\phi^*(u)) - \exp(\phi^*(0)), \quad (9.1)$$

so that  $N(u)$  is even function and  $N(0) = 0$ .

An Orlicz space on the source probability triple generated by the function  $N_\phi(\vec{u})$  will be denoted as ordinary by  $L(N_\phi)$  and the correspondent norm by  $\|\xi\|_{L(N_\phi)}$ .

**Theorem 9.1.** Both the norms  $\|\xi\|_{L(N_\phi)}$  and  $\|\xi\|_{B(\phi)}$  are equivalent on the subspace of the centered random vectors:

$$\exists C_{5,6} = C_{5,6}(d, \phi) \in (0, \infty), \quad 0 < C_5 < C_6 < \infty$$

such that for all the  $d$  - dimensional random vectors  $\xi : \mathbf{E}\xi = 0$

$$C_5 \|\xi\|_{L(N_\phi)} \leq \|\xi\|_{B(\phi)} \leq C_6 \|\xi\|_{L(N_\phi)}. \quad (9.2)$$

**Proof** is at the same as in the one - dimensional case, see [10]. Suppose first of all  $\mathbf{E}\xi = 0$  and  $\|\xi\|_{L(N_\phi)} = 1/2$ . Then

$$\mathbf{E}N_\phi(\xi) \leq 1.$$

The Tchebychev's inequality allows us to estimate the tail behavior of the r.v.  $\xi$  as follows:

$$U(\vec{\xi}, \vec{x}) \leq \frac{1}{N_\phi(\vec{x})} \leq \exp(-\phi^*(c \vec{x})), \quad |x| \geq 1.$$

The case  $|x| \leq 1$  is obvious. The right - hand side of bilateral inequality (9.2) follows immediately from theorem 6.1.

Let us prove now the inverse estimate. Suppose for instance

$$\mathbf{E}e^{(\lambda, \xi)} \leq e^{\phi(\lambda)}, \quad \lambda \in R_+^d.$$

Then  $\mathbf{E}\vec{\xi} = 0$  and

$$U(\vec{\xi}, \vec{x}) \leq e^{-\phi^*(\vec{x})}, \quad \vec{x} > 0.$$

We have analogously to the preprint [15]

$$\mathbf{E}N(0.5 \vec{\xi}) \leq C_1 + C_2 \int_{R_+^d} \exp(\phi^*(0.5 x) - \phi^*(x)) dx < \infty,$$

therefore  $\vec{\xi} \in L(N_\phi)$ .

## 10 Concluding remarks.

### A. Another approach to the problem of description of natural function.

Recall the statement of problem, see fourth section. Given a function  $\phi(\cdot)$  from the set  $Y = Y(R^d)$ . Question: under what additional conditions there exists a mean zero random vector  $\xi$  which may be defined on the appropriate probability space such that

$$\forall \lambda \in R^d \Rightarrow \mathbf{E}e^{(\lambda, \xi)} = e^{\phi(\lambda)}. \quad (10.1)$$

Obviously, the function  $\lambda \rightarrow e^{\phi(\lambda)}$  may be analytically continued on the whole complex space  $\mathbf{C}^d$  such that

$$e^{\phi(i \cdot \vec{t})} = \mathbf{E}e^{i(\vec{t}, \vec{\xi})}, \quad i^2 = -1, \quad \vec{t} \in R^d, \quad (10.2)$$

so that  $\exp\{\phi(i \cdot \vec{t})\}$  is a characteristic function of some  $d$  – dimensional random vector.

### B. Absolutely even functions.

The function  $f : R^d \rightarrow R$  is said to be absolutely even, in  $\forall \vec{\epsilon}, \vec{x}$

$$f(\vec{\epsilon} \otimes \vec{x}) = f(\vec{x}). \quad (10.3)$$

In other words, it is even function separately arbitrary variable by fixed others. For instance, the function of a form

$$f(x, y) = g(x, y) + g(x, -y) + g(-x, y) + g(-x, -y)$$

as well as

$$f_1(x, y) = \max[g(x, y), g(x, -y), g(-x, y), g(-x, -y)]$$

are absolutely even; here  $d = 2$  and  $g : R^2 \rightarrow R$ .

It is interest to note by our opinion that the natural function for any random vector is absolutely even.

### C. Conjugate and associate spaces.

Since the introduced spaces  $B(\vec{\phi})$  coincide with Orlicz spaces, see the 9<sup>th</sup> section, the study of its conjugate and associate spaces may be provided likewise as in monographs [17], [18].

See also [15].

## References

- [1] BENNET C., SHARPLEY R. *Interpolation of operators*. Orlando, Academic Press Inc., (1988).
- [2] BULDYGIN V.V., KOZACHENKO YU. V. *Subgaussian random vectors and theorem of Levi - Bakster*. Theory Probab. Appl., (1986), V.31 Issue 3, 607-609, (in Russian).
- [3] BULDYGIN V.V., KOZACHENKO YU.V. *About subgaussian random variables*. Ukrainian Math. Journal, 1980, 32, No 6, 723-730.
- [4] BULDYGIN V.V., KOZACHENKO YU.V. *Metric Characterization of Random Variables and Random Processes*. 1998, Translations of Mathematics Monograph, AMS, v.188.
- [5] FELLER W. *An introduction to probability theory and its applications*. Volume II, John Wiley and Sons, New York (1971).
- [6] FROLOV A.S., TCHENTZOV N.N. *On the calculation by the Monte-Carlo method definite integrals depending on the parameters*. Journal of Computational Mathematics and Mathematical Physics, (1962), V. 2, Issue 4, p. 714-718 (in Russian).
- [7] GRIGORJEVA M.L., OSTROVSKY E.I. *Calculation of Integrals on discontinuous Functions by means of depending trials method*. Journal of Computational Mathematics and Mathematical Physics, (1996), V. 36, Issue 12, p. 28-39 (in Russian).
- [8] RICHARD JOZSA AND GRAEME MITCHISON. *Entropy, subentropy and the elementary symmetric functions*. arXiv:1310.6629v1 [quant-ph] 24 Oct 2013
- [9] KAHANE J.P. *Propriétés locales des fonctions à séries de Fourier aléatoires*. Studia Math. (1960), 19, No 1, 1-25.
- [10] KOZACHENKO YU. V., OSTROVSKY E.I. (1985). *The Banach Spaces of random Variables of subgaussian Type*. Theory of Probab. and Math. Stat. (in Russian). Kiev, KSU, **32**, 43-57.

- [11] KRASNOSEL'SKII, M.A.; RUTICKII, YA.B. (1961). *Convex Functions and Orlicz Spaces*. Groningen: P.Noordhoff Ltd.
- [12] KURBANMURADOV O., SABELFELD K. (2007). *Exponential bounds for the probability deviation of sums of random fields*. Preprint. Weierstraß - Institut für Angewandte Analysis und Stochastik (WIAS), ISSN 09468633, p. 1-16.
- [13] OSTROVSKY E.I. (1999). *Exponential estimations for Random Fields and its applications*, (in Russian). Moscow-Obninsk, OINPE.
- [14] OSTROVSKY E.I. (1982). *Generalization of Buldygin-Kozatchenko norms and CLT in Banach space*. Theory Probab. Appl., **27**, V.3, p. 617-619, (in Russian).
- [15] OSTROVSKY E. AND SIROTA L. *Multidimensional probabilistic rearrangement invariant spaces: a new approach*. arXiv:1202.3130v1 [math.PR] 14 Feb 2012
- [16] PROKHOROV YU. V. *Multivariate distributions: Inequalities and limit theorems*. Journal of Soviet Mathematics, September 1974, Volume 2, Issue 5, pp 475-488.
- [17] RAO M.M., REN Z.D. *Theory of Orlicz Spaces*. Marcel Dekker Inc., 1991. New York, Basel, Hong Kong.
- [18] RAO M.M., REN Z.D. *Applications of Orlicz Spaces*. Marcel Dekker Inc., 2002. New York, Basel, Hong Kong.
- [19] SACHKOV V.N. *Combinatorial methods in discrete mathematics*. (1996), Moscow, MCCE.